Very I - favorable space

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The following game was introduced by

P. Daniels, K. Kunen and H. Zhou

On the open-open game, Fund. Math. 145 (1994), no. 3, 205 - 220.

Two players playing on a topological space X.

- Player I choosing a non-empty open set $U \subseteq X$.
- Player II should choosing a non-empty open set $V \subseteq U$
- Player I wins if the union of all open sets which have been chosen by Player II is dense in X

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Let X be a topological space equipped with a topology $\mathcal{T}$.

A space X is called I-*favorable* if player I has a winning strategy. This means that there exists a function $\sigma : \bigcup\{\mathcal{T}^n : n \geqslant 0\} \to \mathcal{T}$ such that for each game

$$\sigma(\varnothing), B_0, \sigma(B_0), B_1, \sigma(B_0, B_1), B_2, \ldots, B_n, \sigma(B_0, \ldots, B_n), B_{n+1}, \ldots$$

the union $\bigcup_{n \geqslant 0} B_n$ is dense in X,

where $\emptyset \neq \sigma(\varnothing) \in \mathcal{T}$ and $B_{k+1} \subset \sigma(B_0, B_1, .., B_k) \neq \emptyset$ and $\emptyset \neq B_k \in \mathcal{T}$ for $k \geqslant 0$.

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A family $\mathcal{C} \subset [\mathcal{T}]^{\leqslant\omega}$ is said to be a *club* if:

(i) $\mathcal{C}$ is closed under increasing ω-chains, i.e., if $C_1 \subset C_2 \subset \ldots$ is an increasing ω-chain from $\mathcal{C}$, then $\bigcup_{n\geqslant 1} C_n \in \mathcal{C}$;

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Definition

Let $\mathcal{T}$ be topology on X we write $C \subset_c \mathcal{T}$ if for any nonempty $V \in \mathcal{T}$ there exists $W \in C$ such that if $U \in C$ and $U \subset W$, then $U \cap V \neq \varnothing$.

Theorem (Daniels-Kunen-Zhou 1994)

Topological space X is I-favorable if and only if the family

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Definition (Daniels-Kunen-Zhou 1994)

A space X is called *very* I-*favorable* if the family

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where $\mathcal{T}$ is a topology on X and $\mathcal{P} \subset_v \mathcal{T}$ means that

for any $\mathcal{S} \subset \mathcal{P}$ and $x \notin \operatorname{cl}_X \bigcup \mathcal{S}$, there exists $W \in \mathcal{P}$ such that $x \in W$ and $W \cap \bigcup \mathcal{S} = \emptyset$.

It is easily seen that $\mathcal{P} \subset_v \mathcal{T}$ implies $\mathcal{P} \subset_c \mathcal{T}$.

$\beta\omega$ is an example of I-favorable space and not very I-favorable space.

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Let X be a topological space equipped with a topology $\mathcal{T}$ and $\mathcal{Q} \subset \mathcal{T}$.

Suppose that there exists a function $\sigma : \bigcup\{\mathcal{Q}^n : n \geqslant 0\} \to \mathcal{Q}$ such that

if $B_0, B_1, \ldots$ is a sequence of non-empty elements of $\mathcal{Q}$ with $B_0 \subset \sigma(\varnothing)$ and $B_{n+1} \subset \sigma((B_0, B_1, \ldots, B_n))$ for all $n \in \omega$, then $\{B_n : n \in \omega\} \cup \{\sigma((B_0, B_1, \ldots, B_n)) : n \in \omega\} \subset_v \mathcal{Q}$.

The function σ is called a *strong winning strategy with respect to* $\mathcal{Q}$. If $\mathcal{Q} = \mathcal{T}$, σ is called a strong winning strategy.

It is clear that if σ is strong winning strategy, then it is a winning strategy for player I in the open-open game.

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Proposition

Let X be a topological space and $\mathcal{Q} \subset \mathcal{T}$ be a family closed under finite intersection. Then there is a strong winning strategy $\sigma : \bigcup\{\mathcal{Q}^n : n \geqslant 0\} \to \mathcal{Q}$ with respect to $\mathcal{Q}$ if and only if the family $\{\mathcal{P} \in [\mathcal{Q}]^{\leqslant\omega} : \mathcal{P} \subset_v \mathcal{Q}\}$ contains a club $\mathcal{C}$ such that every $A \in \mathcal{C}$ is closed under finite intersections.

Proposition

If there exists a base $\mathcal{B}$ of X such that the family $\{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega} : \mathcal{P} \subset_v \mathcal{B}\}$ contains a club, then the family $\{\mathcal{P} \in [\mathcal{T}]^{\leqslant\omega} : \mathcal{P} \subset_v \mathcal{T}\}$ contains a club too, where $\mathcal{T}$ is topology on X.

If X is a completely regular space, then Σ_X denotes the collection of all co-zero sets in X.

Corollary

Let X be a completely regular space and $\mathcal{B} \subset \Sigma_X$ a base for X. If $\{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega} : \mathcal{P} \subset_v \mathcal{B}\}$ contains a club, then the family $\{\mathcal{P} \in [\Sigma_X]^{\leqslant\omega} : \mathcal{P} \subset_v \Sigma_X\}$ contains a club too.

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An inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is said to be a σ-complete, whenever Σ is σ-complete and for every chain $\{\gamma_n : n \in \omega\} \subseteq \Sigma$, such that $\gamma = \sup\{\gamma_n : n \in \omega\} \in \Sigma$, there holds

$$X_\gamma = \varprojlim\{X_{\gamma_n}, \pi^{\gamma_{n+1}}_{\gamma_n}\}.$$

A continuous surjection is called *skeletal* whenever for any non-empty open sets $U \subseteq X$ the closure of $f[U]$ has non-empty interior.

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Theorem (Sz. Plewik and me 2008)

Let X be compact space X is a I-favorable , iff

$$X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\},$$

where $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a σ-complete inverse system, all spaces X_σ are compact and metrizable, and all bonding maps π^σ_ϱ are skeletal.

Proposition(Sz. Plewik and me 2008)

If X is a I-favorable completely regular space then X can be dense embeding into $Y = \varprojlim\{Y_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\{Y_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a σ-complete inverse system, consisting of separable metric spaces, and all bonding maps π^σ_ϱ are skeletal.

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We say that a space X is *an almost limit* of the inverse system $S = \{X_\sigma, \pi^\sigma_\varrho, \Gamma\}$, if X can be embedded in $\varprojlim S$ such that $\pi_\sigma(X) = X_\sigma$ for each $\sigma \in \Gamma$.

We denote this by $X = \mathrm{a} - \varprojlim S$, and it implies that X is a dense subset of $\varprojlim S$. A completely regular space X is *skeletally generated* if X is the almost limit of a σ-complete inverse system $S = \{X_\sigma, \pi^\sigma_\varrho, \Gamma\}$ consisting of separable metric spaces X_σ and skeletal surjective bonding maps π^σ_ϱ.

Theorem (V.Valov 2010)

A completely regular space X is skeletally generated if and only if X is I-favorable

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T. Byczkowski and R. Pol (1976) introduced nearly open sets and nearly open maps as follows. A subset A of a topological space X is *nearly open* if $A \subseteq \operatorname{Int} \operatorname{cl} A$.

A map is *nearly open* if the image of every open subset is nearly open. Continuous nearly open maps were called d-*open* by M. Tkachenko (1981). Obviously, every d-open map is skeletal.

Prposition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a continuous function. Then the following conditions are equivalent:

1. f is d-open;i.e. $f(U) \subset \operatorname{Int}_Y \operatorname{cl}_Y f(U)$ for every open subset $U \subset X$;
2. $\operatorname{cl}_X f^{-1}(V) = f^{-1}(\operatorname{cl}_Y V)$ for any open $V \subset Y$;
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1. f is d-open;i.e. $f(U) \subset \operatorname{Int}_Y \operatorname{cl}_Y f(U)$ for every open subset $U \subset X$;
2. $\operatorname{cl}_X f^{-1}(V) = f^{-1}(\operatorname{cl}_Y V)$ for any open $V \subset Y$;
3. $\{f^{-1}(V) : V \in \mathcal{T}_Y\} \subset_v \mathcal{T}_X$.

T. Byczkowski and R. Pol (1976) introduced nearly open sets and nearly open maps as follows. A subset A of a topological space X is *nearly open* if $A \subseteq \operatorname{Int} \operatorname{cl} A$.

A map is *nearly open* if the image of every open subset is nearly open. Continuous nearly open maps were called d-*open* by M. Tkachenko (1981). Obviously, every d-open map is skeletal.

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A completely regular space X is d-*openly generated* if X is the almost limit of a σ-complete inverse system $S = \{X_\sigma, \pi^\sigma_\varrho, \Gamma\}$ consisting of separable metric spaces X_σ and d-open surjective bonding maps π^σ_ϱ.

Theorem

A Hausdorff space X is very I-favorable if and only if $X = \mathrm{a} - \varprojlim S$, where $S = \{X_A, q^A_B, \mathcal{C}\}$ is a σ-complete inverse system such that all X_A are (not-necessarily Hausdorff) spaces with countable weight and the bonding maps q^A_B are d-open and onto.

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Theorem

A completely regular space X is very I-favorable with respect to the co-zero sets if and only if X is d-openly generated.

Recall that a normal space is called perfectly normal if every open set is a co-zero set.

Corollary

Every perfectly normal very I-favorable space is d-openly generated.

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We say that a space $X \subset Y$ is regularly embedded in Y is there exists a function $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_Y$ satisfying the following conditions for any $U, V \in \mathcal{T}_X$:

- $\mathrm{e}(\emptyset) = \emptyset$;
- $\mathrm{e}(U) \cap X = U$;
- $\mathrm{e}(U) \cap \mathrm{e}(V) = \emptyset$ provided $U \cap V = \emptyset$.

Theorem

A completely regular space is very I*-favorable with respect to the co-zero sets if and only if every* C^**-embedding of* X *in any Tychonoff space* Y *is regular.*

We say that a space $X \subset Y$ is regularly embedded in Y is there exists a function $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_Y$ satisfying the following conditions for any $U, V \in \mathcal{T}_X$:

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Theorem

A completely regular space is very I-favorable with respect to the co-zero sets if and only if every C^*-embedding of X in any Tychonoff space Y is regular.

A compact Hausdorff space X is *openly generated* if X is the limit of a σ-complete inverse system $S = \{X_\sigma, \pi^\sigma_\varrho, \Gamma\}$ consisting of compact metric spaces X_σ and open surjective bonding maps π^σ_ϱ.

Theorem

A compact Hausdorff space is very I*-favorable with respect to the co-zero sets if and only if* X *is openly generated.*

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Theorem

A compact Hausdorff space is very I*-favorable with respect to the co-zero sets if and only if* X *is openly generated.*

We say that a topological space X is *perfectly κ-normal* if for every open and disjoint subset U, V there are open F_σ subset W_U, W_V with $W_U \cap W_V = \emptyset$ and $U \subset W_U$ and $V \subset W_V$.

Proposition

If a normal perfectly κ-normal space is a continuous image of a very I-favorable space under a perfect map, then X is d-openly generated.

Proposition

If the image of a compact Hausdorff very I-favorable space under a continuous map is perfectly κ-normal, then X is openly generated.

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